Igor Pak, UCLA

June 2, 2026

Vanishing of Schubert coefficients

9th WACT

University of Copenhagen

Main Theorem (P.-Robichaux, 2025)

Positivity of Schubert coefficients can be decided in probabilistic polynomial time

Main Theorem (P.-Robichaux, 2025)

Positivity of Schubert coefficients can be decided in probabilistic polynomial time

What?

Why?

Goals of the talk

1) *What is this about?*

2) *Why should one care?*

3) *How to prove this?*

Structure constants

$\mathcal{A} \leftarrow$ set of combinatorial objects

$\mathcal{R} = \mathbb{Z}\langle e_\alpha : \alpha \in \mathcal{A}\rangle \leftarrow$ ring with a linear basis

$$e_\alpha \cdot e_\beta = \sum_{\gamma \in \mathcal{A}} c^\gamma_{\alpha\beta}\, e_\gamma \leftarrow \text{structure constants}$$

Main problem:

Give a *combinatorial interpretation* for $\{c^\gamma_{\alpha\beta}\}$

Two more problems:

$[c^\gamma_{\alpha\beta} =^? 0] \leftarrow$ *the vanishing problem*

$[c^\gamma_{\alpha\beta} >^? 0] \leftarrow$ *the positivity problem*

Littlewood–Richardson coefficients

$\Lambda = \mathbb{Z}\langle s_\lambda \rangle \;\leftarrow\;$ symmetric functions with *Schur functions* as a basis

$c^\lambda_{\mu\nu} = \left|\mathrm{LR}(\lambda/\mu,\nu)\right| \in \mathbb{N} \;\leftarrow\;$ LR coefficients count the number of LR tableaux

$$s_\lambda = \frac{\det\left(\left(x_i^{\lambda_j+n-j}\right)_{i,j\in[n]}\right)}{\det\left(\left(x_i^{n-j}\right)_{i,j\in[n]}\right)}$$

$$s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu\nu}\, s_\lambda$$

irreducible $\mathrm{GL}(n,\mathbb{C})$ *characters*

$$\begin{array}{|c|c|c|c|c|c|}
\cline{4-6}
\multicolumn{3}{c|}{} & 1 & 1 & 1 \\
\cline{3-6}
\multicolumn{2}{c|}{} & 1 & 2 & \multicolumn{2}{c}{} \\
\cline{2-4}
\multicolumn{1}{c|}{} & 2 & 2 & 3 & \multicolumn{2}{c}{} \\
\cline{1-4}
3 & 4 & \multicolumn{4}{c}{} \\
\cline{1-2}
\end{array}$$

$\in \mathrm{LR}(6442/321,\, 4321)$

Kronecker coefficients

$\chi^\lambda, \chi^\mu, \chi^\nu \leftarrow$ irreducible S_n characters

$\chi^\lambda \cdot \chi^\mu = \sum_\nu g(\lambda,\mu,\nu)\chi^\nu \leftarrow$ *Kronecker coefficients*

Introduced by Murnaghan (1938)
Generalize LR coefficients
Key role in GCT

Positivity Problems and Conjectures in Algebraic Combinatorics

Richard P. Stanley

Schubert coefficients

$$\mathbb{Z}[x_1, x_2, \ldots] = \mathbb{Z}\langle \mathfrak{S}_w : w \in S_\infty \rangle \quad \leftarrow \quad \textit{Schubert polynomials}$$

$$\mathfrak{S}_u \cdot \mathfrak{S}_v = \sum_{w \in S_\infty} c^w_{u,v} \mathfrak{S}_w \quad \leftarrow \quad \textit{Schubert coefficients}$$

Introduced by Schubert (1880s)
Reintroduced by Lascoux and Schützenberger (1982)
Represent cohomology classes in complete flag varieties

MATHEMATICAL PROBLEMS

BY DAVID HILBERT

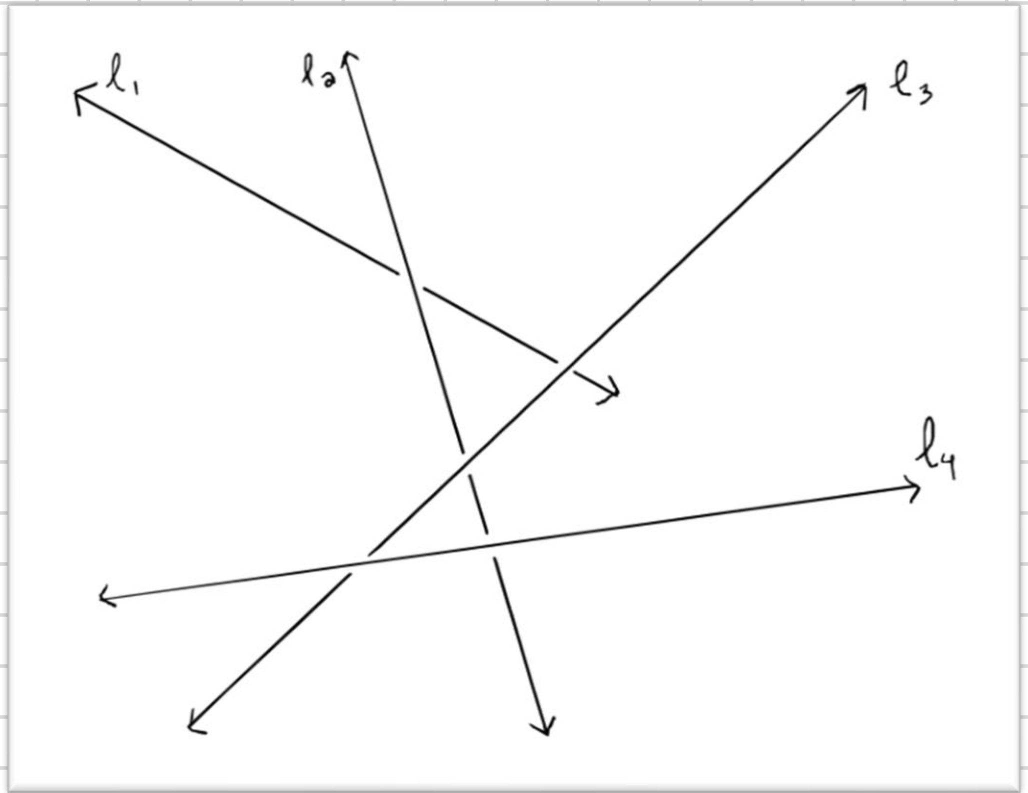
Positivity Problems and Conjectures in Algebraic Combinatorics

Richard P. Stanley

Schubert coefficients

Hilbert's 15th Problem (1900)

Question: *How many lines meet four given lines in general position?*



Answer: 2

Schubert coefficients

Schubert polynomial $\mathfrak{S}_w \in \mathbb{N}[x_1, x_2, \ldots]$ is defined as

$$\mathfrak{S}_w(\boldsymbol{x}) := \sum_{H \in \mathrm{RC}(w)} \boldsymbol{x}^H \quad \text{where} \quad \boldsymbol{x}^H := \prod_{(i,j)\,:\,H(i,j)=\boxplus} x_i\,.$$

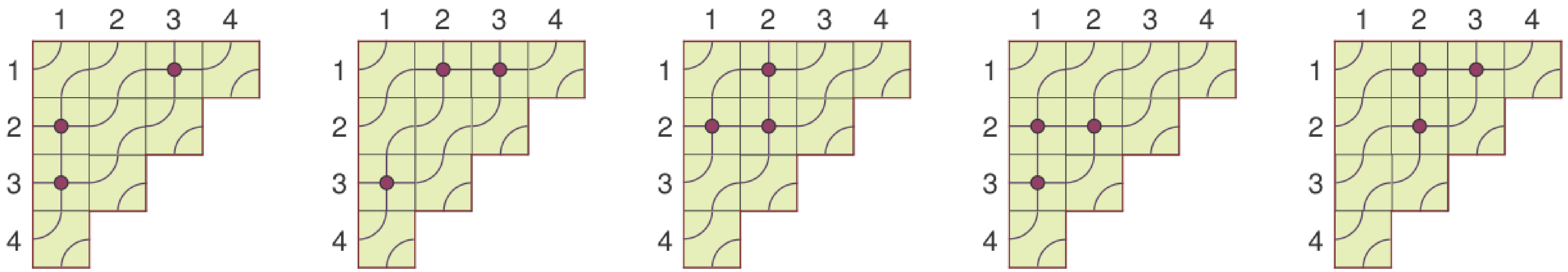


FIGURE 1.2. Graphs in RC(1432) and the corresponding Schubert polynomial $\mathfrak{S}_{1432} = x_1x_2x_3 + x_1^2x_3 + x_1x_2^2 + x_2^2x_3 + x_1^2x_2$ with monomials in this order.

What happened? A personal story

Kronecker and Schubert Coefficients DO NOT *have a combinatorial interpretation*

2012-20
Combinatorial Interpretation $\longleftarrow$ # integer points in polytopes
Dozens of papers on Kronecker coefficients & integer points

Theorem [Nguyen–P., CCC'17]
For $P, Q \in \mathbb{Q}^3$, computing $|(P \smallsetminus Q) \downarrow_x \cap \mathbb{Z}|$ is #P-complete.

What happened? A personal story

Kronecker and Schubert Coefficients DO NOT *have a combinatorial interpretation*

2012-20
Combinatorial Interpretation $\longleftarrow$ # integer points in polytopes
Dozens of papers on Kronecker coefficients & integer points

Conclusion: *This is a wrong definition!*

What happened? A personal story

Kronecker and Schubert Coefficients DO NOT *have a combinatorial interpretation*

2020-24

Combinatorial Interpretation $\leftarrow$ #P

First "not in #P" tools with C. Ikenmeyer, S.H. Chan

Conclusion: *Vanishing is the key to the problem!*

Mini Theory [Ikenmeyer–P., FOCS'22]

- $(f-g)^2 \notin \#\mathrm{P}$ unless $\mathrm{PH} = \Sigma_2$ where $f \leftarrow \#\mathsf{3SAT}(\Phi)$, $g \leftarrow \#\mathsf{3SAT}(\Psi)$
- *Motzkin polynomial* $f^2g^4 + f^4g^2 + 1 - 3f^2g^2 \notin \#\mathrm{P}$ unless $\mathsf{UP} = \mathsf{coUP}$
- general closure properties of polynomials and semi-algebraic sets

Theorem [Ikenmeyer–P.-Panova, SODA'23]

- $\left|\chi^\lambda(\mu)\right|^2 \notin \#\mathrm{P}$ unless $\mathrm{PH} = \Sigma_2$
- $\left\{(\lambda,\mu) : \chi^\lambda(\mu) = 0\right\}$ is $\mathrm{C}_=\mathrm{P}$-complete

Theorem [Chan–P., STOC'24]

- defect of the *Alexandrov–Fenchel inequality* $\notin \#\mathrm{P}$ unless PH collapses
- {equality cases of the AF inequality} $\notin$ PH unless PH collapses

Main problems, updated

Conjecture [P.'22] Kronecker Positivity $\notin$ PH

$\Longrightarrow$ Kronecker coefficients $\notin$ #P (no combinatorial interpretation)

Conjecture [P.'22] Schubert Positivity $\notin$ PH

$\Longrightarrow$ Schubert coefficients $\notin$ #P (no combinatorial interpretation)

Kronecker positivity

Conjecture [P.'22] Kronecker Positivity $\notin$ PH
$\implies$ Kronecker coefficients $\notin$ #P (no combinatorial interpretation)

Theorem [Ikenmeyer–Mulmuley–Walter'17]
Kronecker Positivity is NP-complete

Theorem [Bravyi–Chowdhury–Gosset–Havlíček–Zhu'24]
Kronecker Positivity is in QMA

Schubert positivity

Conjecture [P.'22] $\textsc{Schubert Positivity} \notin \mathsf{PH}$

$\implies$ Schubert coefficients $\notin \#\mathsf{P}$ (no combinatorial interpretation)

FALSE

Conjecture [Adve–Robichaux–Yong'22] $\textsc{Schubert Positivity}$ is coNP-hard

FALSE

Conjecture [P.–Robichaux'24] $\textsc{Schubert Positivity}$ is NP-hard

FALSE

(unless PH collapses)

Motivation

$\mathcal{F}_n = \{1 = U_0 \subset U_1 \subset U_2 \subset \cdots U_n = \mathbb{C}^n\}$

Schubert Positivity $\left[c^u_{vw} >^? 0\right]$ is equivalent to:

Input: $n \times n$ integral matrices (a_{ij}), (b_{ij}), (c_{ij})

Decide: $\forall U_\bullet, V_\bullet \in \mathcal{F}_n$ s.t. $U_i \cap V_{n-i} = \{\mathbf{0}\}$ for all $1 \le i < n$,

$\exists W_\bullet \in \mathcal{F}_n$ s.t. $\dim(W_i \cap E_j) = a_{ij}$, $\dim(W_i \cap U_j) \ge b_{ij}$

and $\dim(W_i \cap V_j) \ge c_{ij}$, for all $1 \le i, j \le n$

[Knutson, ICM paper, 2022]

"For applications (including real-world engineering applications) it is more important to know that [Schubert] structure constant is positive, than it is to know its actual value."

Prior work: LR coefficients

Theorem [De Loera–McAllister'06, Mulmuley–Narayanan–Sohoni'12]

LR Vanishing $\in$ P

Two step proof:

(1) $c^{\lambda}_{\mu\nu} = \left|\mathrm{GT}(\lambda/\mu,\nu) \cap \mathbb{Z}^d\right| \leftarrow$ number of integer points in *Gelfand–Tsetlin polytopes*

(2) $c^{\lambda}_{\mu\nu} > 0 \iff c^{k\lambda}_{k\mu k\nu} > 0$ for any k $\leftarrow$ *Knutson–Tao saturation theorem* (1999)

saturation property

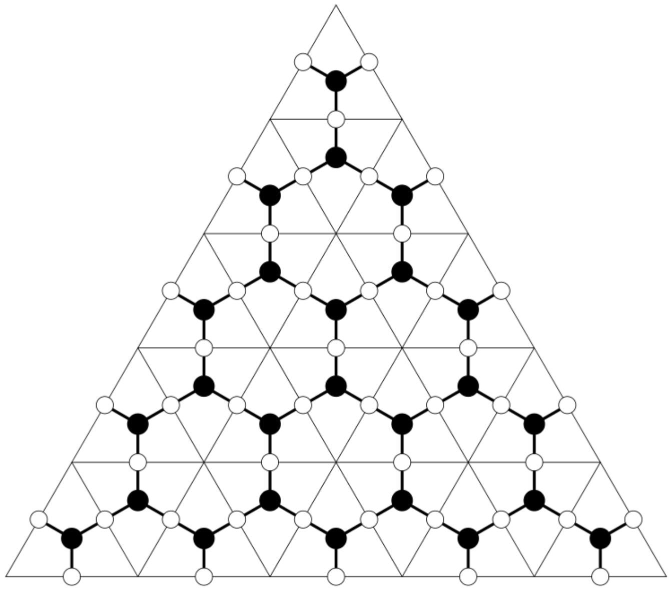
$$c^{\lambda}_{\mu,\nu} > 0 \iff c^{N\lambda}_{N\mu,N\nu} > 0 \;\; \textit{for any} \;\; N \geq 1.$$

Prior work: LR coefficients

Theorem [Bürgisser–Ikenmeyer'13]

LR Positivity can be decided in $O(\ell(\lambda)^3 \log \lambda_1)$ arithmetic operations.

Note: Proof uses network flow analysis in modified Knutson–Tao hives.



No saturation for Schubert coefficients

Theorem [P.–Robichaux'26, P.–Slonim'26, former *Kirillov's conjecture*'04]

Saturation fails for Schubert coefficients.

Here $u, v, w \in S_n$ are given by their Lehmer code, and have two descents.

saturation property

$$c^{w}_{u,v} > 0 \quad \Longleftrightarrow \quad c^{N*w}_{N*u,N*v} > 0 \ \textit{ for any } \ N \geq 1.$$

Prior work

Many necessary/sufficient conditions for Schubert Positivity:

- Lascoux–Schützenberger the *number of descents condition* (1982)
- strong Bruhat order condition, ibid.
- Knutson's descent cycling condition (2001)
- Billey–Vakil permutation array condition (2008)
- St. Dizier–Yong's fillings of Rothe diagrams condition (2022)
- Hardt–Wallach's empty rows in Rothe diagrams condition (2024)
- Purbhoo's root game conditions (2006)
- Knutson and Zinn-Justin's several (!) tiling conditions (2017, 2023)

Schubert coefficients

$4 + 2\omega < 8.75$

Main Theorem [P.–Robichaux'25]:

SCHUBERT POSITIVITY is in RP.

More precisely, for all $k \geq 3$ and $\varepsilon > 0$, there is a probabilistic algorithm which inputs elements $u, v, w \in S_n$ and after $O\big(n^{8.75} \log \frac{1}{\varepsilon}\big)$ arithmetic operations outputs either:

- $c > 0$, which holds with probability $p = 1$, or
- $c^w_{u,v} = 0$, which holds with probability $p > 1 - \varepsilon$.

Note: The timing is comparable with LP algorithm for LR POSITIVITY.

Bürgisser–Ikenmeyer network flow algorithm for LR POSITIVITY is *much* faster (2013).

Structure constants for Schur P-functions

$P_\lambda(\boldsymbol{x}) = P_\lambda(\boldsymbol{x}, -1) \in \mathbb{Z}[p_1, p_3, p_5, \ldots] \leftarrow$ Schur P-functions (Schur, 1911)

correspond to spin (projective) irreducible characters of S_n

$P_\mu \cdot P_\nu = \sum_\lambda f_{\mu\nu}^\lambda P_\lambda \leftarrow$ structure constants

Notes:

- $f_{\mu\nu}^\lambda =$ number of certain shifted tableaux (Serrano'09, Cho'12, Nguyen'22, etc.)
- saturation fails (Robichaux–Yadav–Yong'22)
- corresponds to Schubert structure constants for Grassmannians in types B/D

Structure constants for Schur P-functions

$P_\lambda(\boldsymbol{x}) = P_\lambda(\boldsymbol{x}, -1) \in \mathbb{Z}[p_1, p_3, p_5, \ldots] \leftarrow$ Schur P-functions (Schur, 1911)

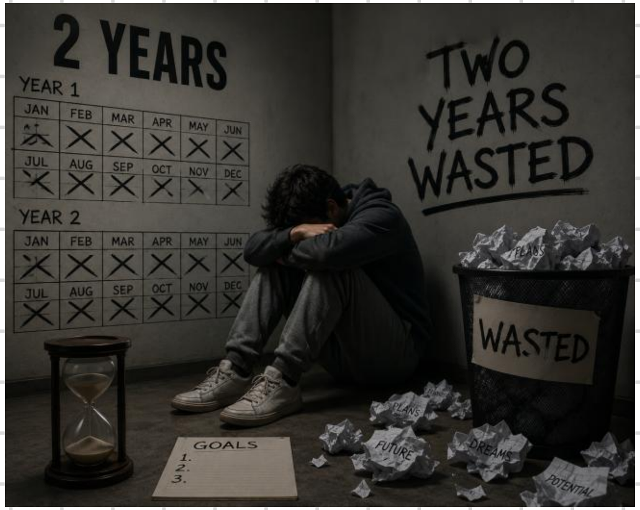
correspond to spin (projective) irreducible characters of S_n

$P_\mu \cdot P_\nu = \sum_\lambda f^\lambda_{\mu\nu} P_\lambda \leftarrow$ structure constants

Conjecture: Schur P Positivity $\left[f^\lambda_{\mu\nu} >^? 0\right] \in \mathrm{P}$

Corollary [P.–Robichaux'25]: $\left[f^\lambda_{\mu\nu} >^? 0\right] \in \mathrm{RP}$

Evolution of our work



Conjecture [P.'22] SCHUBERT POSITIVITY $\notin$ PH

$\Longrightarrow$ Schubert coefficients $\notin$ #P (no combinatorial interpretation)

Evolution of our work

1. (Dec 3, 2024): Schubert Positivity is in AM assuming GRH (*Proc. STOC'25*)

GRH = Generalized Riemann Hypothesis

Additionally, Schubert Positivity is in $\mathrm{NP}_{\mathbb{C}} \cap \mathrm{P}_{\mathbb{R}}$ (unconditionally)

1.5 (Feb 20, 2025): uniform proof in all types (type D joint with Speyer)

2. (Apr 3, 2025): Schubert Positivity is in $\mathrm{AM} \cap \mathrm{coAM}$ assuming GRH

(*Forum Sigma*, 2025)

3. (Sep 19, 2025): Schubert Positivity is in RP (unconditionally)

Note: $\mathrm{RP} \subseteq \mathrm{NP} \cap \mathrm{BPP}$

First approach

Hilbert's Nullstellensatz (HN):

Given $f_i \in \mathbb{Z}[x_1, \ldots, x_\ell]$, decide: $\exists \boldsymbol{x} = (x_1, \ldots, x_\ell) \in \mathbb{C}^\ell$ s.t. $f_1(\boldsymbol{x}) = \ldots = f_m(\boldsymbol{x}) = 0$

Theorem [Hilbert's weak Nullstellensatz]

$\neg$HN $\iff$ $\exists g_1, \ldots, g_m \in \mathbb{Z}[x_1, \ldots, x_\ell]$ s.t. $f_1 g_1 + \ldots + f_m g_m = 1$

[Mayr–Meyer'82]: HN is in EXPSPACE

[Kollár'88]: HN is in PSPACE

[Koiran'96]: HN is in AM assuming GRH

First approach

PARAMETRIC HILBERT'S NULLSTELLENSATZ (HNP):

Given $f_i \in \mathbb{Z}(y_1, \ldots, y_k)[x_1, \ldots, x_\ell]$, decide: $\exists \boldsymbol{x} \in \left(\overline{\mathbb{C}[y_1, \ldots, y_k]}\right)^\ell$ s.t. $f_1(\boldsymbol{x}) = \ldots = f_m(\boldsymbol{x}) = 0$

[A+'24]: HNP is in AM assuming GRH

$\{A+\}$:= Ait El Manssour, Balaji, Nosan, Shirmohammadi and Worrell

First approach

Main Lemma [P–Robichaux]: SCHUBERT POSITIVITY reduces to HNP

Proof idea: Rewrite $c^{w}_{u,v} = \#\{X_u(E_\bullet) \cap X_v(F_\bullet) \cap X_{\widetilde{w}}(G_\bullet)\}$ as an instance of HNP. $\square$

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A lifted square formulation for certifiable Schubert calculus ☆

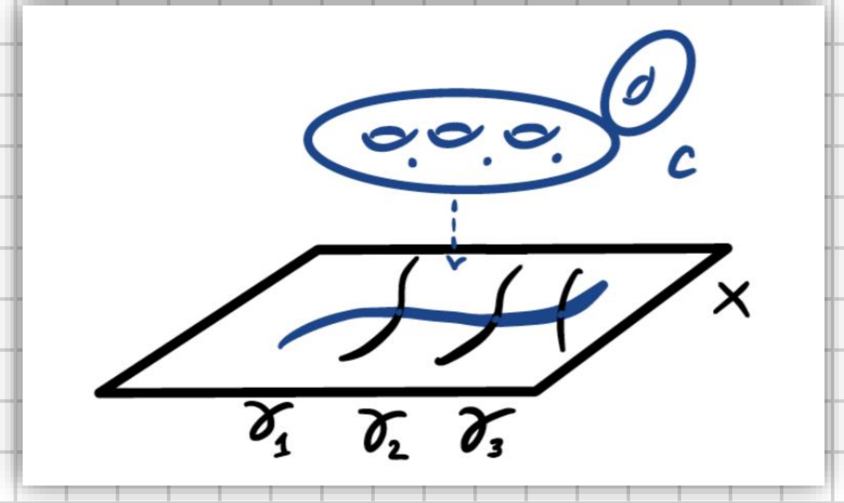
Nickolas Hein [a], Frank Sottile [b]

(2017)

Power of this approach

Theorem [P.–Robichaux–Xu'25]

Deciding if a given genus zero 3-pointed Gromov–Witten invariant is nonzero in the partial flag variety is in AM *assuming the* GRH.



Power of this approach

Theorem [P.–Robichaux–Xu'25]

Deciding if a given genus zero 3-pointed Gromov–Witten invariant is nonzero in the partial flag variety is in AM *assuming the* GRH.

Main Lemma: GW Positivity reduces to HNPE

Complexity Lemma: HNPE is in AM assuming GRH

Note: HNPE extends HNP to polynomials where exponents are also variables

Second approach

$B_n \leftarrow$ group of upper triangular $M \in \mathrm{GL}(n,\mathbb{C})$

$U_n \subset B_n \leftarrow$ group of upper triangular M with 1's on the diagonal

$V_n \leftarrow$ vector space of upper triangular M with 0's on the diagonal

$R_w := V_n \cap w(B_n)^T w^{-1} \leftarrow$ vector space corresponding to $w \in S_n$

$\overline{w} := (w_n, \ldots, w_1)$ for $w = (w_1, \ldots w_n) \in S_n$

Lemma [Purbhoo'06]

Fix $u, v, w \in S_n$ s.t. $\mathrm{inv}(u) + \mathrm{inv}(v) = \mathrm{inv}(w)$. For generic $\rho, \eta, \zeta \in U_n$ we have:

$$c^w_{u,v} > 0 \iff \rho R_u \rho^{-1} + \eta R_v \eta^{-1} + \zeta R_{\overline{w}} \zeta^{-1} = \rho R_u \rho^{-1} \oplus \eta R_v \eta^{-1} \oplus \zeta R_{\overline{w}} \zeta^{-1}$$

Rank condition

Polynomial identity testing

Theorem [Schwarz–Zippel Lemma'79]

Let $\mathbb{F}$ *be a field,* $S \subset \mathbb{F}$ *be a finite set. Let* $Q \in \mathbb{F}[x_1, x_2, \ldots, x_n]$ *be a non-zero polynomial with degree* $d \geq 0$ *over* $\mathbb{F}$. *Then:*

$$\mathbf{P}\left[Q(c_1, c_2, \ldots, c_n) = 0\right] \leq \frac{d}{|S|}$$

where the probability is over random, independent and uniform choices of $c_1, c_2, \ldots, c_n \in S$.

Schwartz–Zippel lemma

From Wikipedia, the free encyclopedia

PROOF

Second approach

Proof idea of the Main Theorem:

Rewrite "generic" condition in the Purbhoo's criterion as an instance of PIT.

Carefully assemble the pieces to obtain the desired probabilistic poly-time algorithm. $\square$

Conjecture [P.–Robichaux'25]:

Schubert Positivity is not in P unless PIT can be decided quasi-polynomial time: $n^{O((\log n)^c)}$

Thank you!